Def: Let $f(x)$ be a function on the interval $a \le x \le b$.
Then for $a < c < b$ the derivative of $f$ at $c$ is defined to be the limit

$$\lim_{h\to 0} \frac{f(c+h)-f(c)}{h}, \text{ if it exists}$$

← difference quotient

We write $f'(c)$ or $\frac{df}{dx}(c)$ for the derivative
$f'(x)$ and $\frac{df}{dx}$ are the derivative of $f$ on $(a,b)$

EX: ① $f(x)=x$

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(\cancel{x}+h)-\cancel{x}}{h} = \lim_{h\to 0}\frac{h}{h} = 1$$

② $f(x) = x^n \quad n \ge 1$ integer

$$(x+h)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \dots + \binom{n}{n-1}xh^{n-1} + \binom{n}{n}h^n$$

$$= \sum_{k=0}^{n}\binom{n}{k}x^{n-k}h^k \qquad \text{Binomial Thm}$$

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$

$\binom{n}{0} = 1, \quad \binom{n}{1} = n$

$$(x+h)^n - x^n = \cancel{x^n} + nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \dots + nxh^{n-1} + h^n - \cancel{x^n}$$

$$\frac{d}{dx}[x^n] = \lim_{h\to 0}\frac{(x+h)^n - x^n}{h} = \lim_{h\to 0}\left(nx^{n-1} + \binom{n}{2}x^{n-2}h + \dots + nxh^{n-2} + h^{n-1}\right) = nx^{n-1}$$

$$\boxed{\frac{d}{dx}[x^n] = nx^{n-1}}$$

③ $f(x) = \sin x$

2 special limits:

$$\lim_{x\to 0}\frac{\sin x}{x} = 1$$

$$\lim_{x\to 0}\frac{(\cos x)-1}{x} = 0$$

$$\frac{d}{dx}[\sin x] = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} = \lim_{h\to 0}\frac{\sin x\cos h + \cos x\sin h - \sin x}{h}$$

$\sin(a \pm b) = \sin a\cos b \pm \cos a \sin b$

$$= \lim_{h\to 0}\left[\sin x\left(\frac{\cosh - 1}{h}\right) + \cos x\left(\frac{\sinh}{h}\right)\right]$$

$$= (\sin x)\lim_{h\to 0}\underbrace{\frac{\cosh-1}{h}}_{=0} + (\cos x)\lim_{h\to 0}\underbrace{\frac{\sinh}{h}}_{=1} = \cos x$$

Properties: $f, g$ are differentiable functions & $c$ is a constant
(it has a derivative)

① $\frac{d}{dx}[cf(x)] = cf'(x)$

② $\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$

③ Product Rule

$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$

④ Quotient Rule

$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2}$

⚠ $g(x) \neq 0$

---

Ex: ① $x^3 \cos x$

$\frac{d}{dx}[x^3 \cos x] = \frac{d}{dx}[x^3]\cos x + x^3 \frac{d}{dx}[\cos x]$

$= 3x^2 \cos x + x^3(-\sin x)$

$= 3x^2 \cos x - x^3 \sin x$

② $x(\sin x)(\cos x)$

$\frac{d}{dx}[x(\sin x)(\cos x)] = \frac{d}{dx}[x](\sin x \cos x) + x\frac{d}{dx}[\sin x \cos x]$

$= \sin x \cos x + x\left[\frac{d}{dx}(\sin x)\cos x + \sin x \frac{d}{dx}(\cos x)\right]$

$= \sin x \cos x + x[\cos^2 x - \sin^2 x]$

$\boxed{\frac{d}{dx}[fgh] = f'gh + fg'h + fgh'}$

③ $\tan x$

$\frac{d}{dx}[\tan x] = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\frac{d}{dx}(\sin x)\cos x - \frac{d}{dx}(\cos x)\sin x}{\cos^2 x}$

$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$

$= \frac{1}{\cos^2 x}$

$= \sec^2 x$

④ $\frac{x\cos x}{x^2-2}$

$\frac{d}{dx}\frac{x\cos x}{x^2-2} = \frac{\frac{d}{dx}(x\cos x)(x^2-2) - \frac{d}{dx}(x^2-2)(x\cos x)}{(x^2-2)^2}$

leave here → $= \frac{(\cos x - x\sin x)(x^2-2) - (2x)(x\cos x)}{(x^2-2)^2}$

$= \frac{x^2\cos x - 2\cos x - x^3\sin x + 2x\sin x - 2x^2\cos x}{(x^2-2)^2}$

$= \frac{-x^2\cos x - 2\cos x - x^3\sin x + 2x\sin x}{(x^2-2)^2}$

$\frac{d}{dx}[\sec x] = \sec x \tan x$

$\frac{d}{dx}[\csc x] = -\csc x \cot x$

$\frac{d}{dx}[\cot x] = -\csc^2 x$

<u>Chain Rule</u>

$f, g$ differentiable & $f \circ g$ is a function (i.e. range($g$) is in the domain of ($f$))

$$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$$

<u>EX</u>: $\frac{d}{dx}[\sin(x^2)] = \cos(x^2) \cdot 2x$

$f = \sin x$

$g = x^2$